

A Full Computation-relevant Topological Dynamics Classification of Elementary Cellular Automata

Martin Schüle, Ruedi Stoop

Institute of Neuroinformatics,

ETH and University of Zurich, 8057 Zurich, Switzerland

Email: `schuelelem@ini.phys.ethz.ch`

Abstract

Cellular automata are both computational and dynamical systems. We give a complete classification of the dynamic behaviour of elementary cellular automata in terms of fundamental dynamic system notions such as sensitivity and chaoticity. We conjecture that elementary cellular automata capable of carrying out "complex" computational tasks are sensitive but not chaotic, i.e. at the "edge of chaos".

1 Introduction

The theory of computation has an exceedingly rich and successful history. Its models of computation, typically based on the Turing machine model, allow to study the capacities of computational systems in great detail [1, 2]. It is however not quite clear how "computations" that occur in nature can be cast in terms of the conventional theory of computation. One would like to ascribe in some sense "computational" capacities to the intricate dynamics of systems found in nature, especially to the dynamics of biological systems. To describe the dynamic behaviour of physical and biological systems, *dynamical system theory* seems a more appropriate framework. Hence, it is a promising approach to study classes of systems or models that comprise in a generic manner both computational and dynamical system aspects. A paradigmatic class of such models are the *cellular automata* (CA).

By definition, CA are discrete dynamical systems acting in a discrete space-time. The state of a CA is specified by the states of the individual cells of the CA, that is by the values taken from a finite set of states associated with the sites of a regular, uniform, infinite lattice. The state of a CA then evolves in discrete time steps according to a rule acting synchronously on the states in a finite neighbourhood of each cell. Despite the simplicity of these rules, CA can exhibit strikingly complex dynamical behaviour. A well-known example

of a CA with intricate dynamics is the so-called *Game of Life*. CA have also been extensively applied as models for a wide variety of physical and biological processes.

While being a class of discrete dynamical systems, CA are, evidently, also a mathematical model of massively parallel computation. Remarkably, already very simple rules can make a CA computationally universal, i.e. capable of carrying out arbitrary computational tasks. It is thus a natural and important question in what regard the computational capacities of CA relate to the dynamical system properties of CA.

We will contribute to a more profound understanding of CA as both computational and dynamical systems by clarifying the dynamical system properties of the most popular and best-studied class of CA, the so-called elementary cellular automata (ECA). Our approach is based on the symbolic dynamics treatment of CA initiated by the seminal paper of Hedlund [3]. The symbolic dynamics theory of CA allows to study the dynamical properties of CA by topological dynamics methods.

Obtaining a dynamical system classification of ECA is part of the long-standing problem in CA theory to characterise the "complexity" seen inherent in CA behaviour. In a series of influential papers, Wolfram studied the dynamical system and statistical properties of CA and devised a classification scheme [4, 5, 6]. According to this scheme, CA behaviour can be divided into the following classes:

- (W1) almost all initial configurations lead to the same fixed point configuration,
- (W2) almost all initial configurations lead to a periodic configuration,
- (W3) almost all initial configurations lead to random looking behaviour,
- (W4) localized structures with complex behaviour emerge.

Wolfram's classification attempt was largely based on simulations of ECA. Since his pioneering work many more classification schemes have been proposed, e.g. by Li et al. [7] or Culik et al. [8]. It is however still an open problem of CA theory to obtain a completely satisfying, formal classification of CA behaviour.

We put forward a complete topological dynamics classification of ECA. Such a classification uses the fundamental notions of dynamics system theory as e.g. sensitivity, chaos, etc. More specifically, it is based on a scheme introduced by Gilman [9] and modified by Kurka [10] which proposes four classes: Equicontinuous CA, CA with some equicontinuous points, sensitive but not positively expansive CA and positively expansive CA. Each one-dimensional CA belongs to exactly one class, but class membership is generally not decidable [10]. We determine for every ECA, as far as we know for the first time, to which class it belongs. We also (re-)derive further properties such as surjectivity and chaoticity of ECA. Taken together this gives a fairly complete picture of the dynamical system properties of ECA.

The paper is organised as follows. In section 2 we introduce one-dimensional CA and ECA formally. In section 3 we give basic notations and definitions of the topological dynamics approach to CA. In section 4 we introduce a scheme that allows to express ECA rules algebraically. This will prove helpful in sections 5 and 6, where we will classify ECA in the topologically dynamics sense of Kurka.

In section 7 we discuss our results.

2 Definition of Elementary Cellular Automata

We start with the definitions of the basic concepts underlying the theory of one-dimensional CA. The *configuration* of a one-dimensional CA is given by the double-infinite sequence $x = (x_i)_{i \in \mathbb{Z}}$ with $x_i \in S$ being elements of the finite set of states $S = \{0, 1, \dots\}$. The configuration space X is the set of all sequences x , i.e. $X = S^{\mathbb{Z}}$. The CA map F , simply called the CA F , is a map $F : X \rightarrow X$ where the *local function* is the map $f : S^{2r+1} \rightarrow S$, $r \geq 1$, with $F(x)_i = f(x_{i-r}, \dots, x_i, \dots, x_{i+r})$. The integer r is called the *radius* of the CA. The iteration of the map F acting on an initial configuration x generates the *orbit* $x, F(x), F^2(x), \dots$ of x . The orbits of all configurations x are a discrete space-time dynamical system also referred to as CA F . Instances of the system can be visualised in so-called *space-time patterns*.

A *spatially periodic* configuration is a configuration which is invariant under translation in space, that is x is periodic if there is $q > 1$ such that $\sigma^q(x) = x$ where $\sigma : X \rightarrow X$ is the *shift map* $\sigma(x)_i = x_{i+1}$. A *temporally periodic* or simply *periodic* configuration x for some CA F is given if $F^n(x) = x$ for some $n > 0$. If $F(x) = x$, x is called a *fixed point*. A configuration x is called *eventually periodic*, if it evolves into a temporally periodic configuration, i.e. if $F^{k+n}(x) = F^k(x)$ for some $k \geq 0$ and $n > 0$. If this holds for any configuration x , the corresponding CA is called *eventually periodic*.

An elementary cellular automaton (ECA) is an one-dimensional CA with two states and “nearest neighbourhood coupling”, that is $S = \{0, 1\}$ and $r = 1$. There are then 256 different possible local functions $f : S^3 \rightarrow S$ with $F(x)_i = f(x_{i-1}, x_i, x_{i+1})$. Local functions are also called *rules* and usually given in form of a *rule table*. An example is:

111	110	101	100	011	010	001	000
0	1	1	0	1	1	1	0

Every ECA rule is, following Wolfram [4], referred to by the sequence of the values of the local function, as given in the rule table, written as a decimal number. In the example above one speaks of ECA rule 110, because 01101110 written as a decimal number equals 110.

3 Topological and Symbolic Dynamics Definitions and Concepts

The framework we use to study the dynamical properties of ECA is given by the symbolic dynamics approach that views the state space $S^{\mathbb{Z}}$ of one-dimensional CA as the Cantor space of symbolic sequences. The topology of the Cantor

space is induced by the metric

$$d_C(x, y) = \sum_{i=-\infty}^{+\infty} \frac{\delta(x_i, y_i)}{2^{|i|}},$$

where $\delta(x_i, y_i)$ is the discrete metric $\delta(x_i, y_i) = \begin{cases} 1, & x_i \neq y_i \\ 0, & x_i = y_i \end{cases}$.

Under this metric the configuration space $S^{\mathbb{Z}}$ is compact, perfect and totally disconnected, i.e. a Cantor space [11]. From now on, the configuration space $S^{\mathbb{Z}}$ endowed with this metric will be referred to as X . The ECA functions F are continuous in X , hence (X, F) is a (discrete) dynamical system.

Now we introduce some key concepts of the topological dynamics treatment of CA. A configuration x is an *equicontinuity point* of CA F if

$$\forall \epsilon > 0, \exists \delta > 0, \forall y \in X : d(x, y) < \delta, \forall n \geq 0 : d(F^n(x), F^n(y)) < \epsilon. \quad (1)$$

If all configurations $x \in X$ are equicontinuity points then the CA is called *equicontinuous*. If there is at least one equicontinuity point, the CA is *almost equicontinuous*.

A CA is *sensitive* (to initial conditions) if

$$\exists \epsilon > 0, \forall x \in X, \forall \delta > 0, \exists y \in X : d(x, y) < \delta, \exists n \geq 0 : d(F^n(x), F^n(y)) \geq \epsilon. \quad (2)$$

A CA is *positively expansive* if

$$\exists \epsilon > 0, \forall x \neq y \in X, \exists n \geq 0 : d(F^n(x), F^n(y)) \geq \epsilon. \quad (3)$$

Positively expansive CA are sensitive [11].

CA as dynamical systems can be classified according to a classification introduced by Gilman [9] and modified by Kurka [10]. Every one-dimensional CA falls exactly in one of the following classes [10]:

- (K1) equicontinuous CA,
- (K2) almost equicontinuous but not equicontinuous CA,
- (K3) sensitive but not positively expansive CA,
- (K4) positively expansive CA.

It has been shown that for one-dimensional CA it is not decidable if a given CA belongs to class (K1), (K2) or (K3)∪(K4), whereas it is open whether the class (K4) is decidable [12]. We will show that, in the case of ECA, it can be determined of each given ECA in which class it belongs.

4 Algebraic Expressions of Elementary Cellular Automata Rules

Here, we devise an algebraic expression scheme for ECA. The main idea is to derive in a consistent way algebraic expressions for the local ECA rules from a

Boolean function form of ECA rules. The algebraic expressions of ECA rules are of use in the next sections. Algebraic expressions of specific ECA rules have been given before, usually of additive ECA rules (e.g. [13]), for example rule 90 is usually given as $F(x)_i = x_{i-1} + x_{i+1} \pmod{2}$ [11]. Other approaches, e.g. by Chua [14], do not yield the same simple polynomial forms as proposed below. The approach taken here was first proposed by the authors in [15]. As far as we know, it was the first time that simple, algebraic expressions for *all* ECA rules were given. Note that Betel and Flocchini used a similar approach in their study on the relationship between Boolean and “fuzzy” cellular automata [16].

The rule tables which define the ECA rules can be regarded as truth tables familiar from propositional logic. Any ECA rule hence corresponds to a *Boolean function*, which can always be expressed as a *disjunctive normal form* (DNF) (or a conjunctive normal form respectively) [17]. The DNF of a Boolean function is a disjunction of clauses, where a clause is a conjunction of Boolean variables. Any ECA rule can thus be expressed as

$$\bigvee_m \bigwedge_{j=-1}^1 (\neg) X_{i+j}^m \quad (4)$$

where X_{i+j} are Boolean variables associated with the states of the cells in the neighbourhood of an ECA. For example the DNF expression of ECA rule 110 reads to:

$$(X_{i-1} \wedge X_i \wedge \neg X_{i+1}) \vee (X_{i-1} \wedge \neg X_i \wedge X_{i+1}) \vee (\neg X_{i-1} \wedge X_i \wedge X_{i+1}) \vee (\neg X_{i-1} \wedge X_i \wedge \neg X_{i+1}) \vee (\neg X_{i-1} \wedge \neg X_i \wedge X_{i+1}).$$

The representation of ECA rules in DNF is well known and has e.g. be studied by Wolfram [18].

We however proceed now further by expressing the Boolean operations (\wedge, \vee, \neg) arithmetically as

$$\begin{aligned} x \wedge y &= xy \\ x \vee y &= x + y - xy \\ \neg x &= 1 - x. \end{aligned} \quad (5)$$

By these operations we attain from the Boolean algebra $(A, \wedge, \vee, \neg, 1, 0)$ the Boolean ring $(R, +, -, \cdot, 1, 0)$. We found it convenient to express the Boolean operations as defined above instead of using the usual (mod 2) operations.

Replacing the Boolean operations in the DNF expressions of ECA rules with their arithmetic counterparts, we obtain for all ECA algebraic expressions, that is Boolean polynomials

$$\sum_{\alpha} p_{\alpha} x^{\alpha}$$

of the form

$$\alpha_0 + \alpha_1 x_{i-1} + \alpha_2 x_i + \alpha_3 x_{i+1} + \alpha_4 x_{i-1} x_i + \alpha_5 x_i x_{i+1} + \alpha_6 x_{i-1} x_{i+1} + \alpha_7 x_{i-1} x_i x_{i+1}. \quad (6)$$

The coefficients $\alpha_i \in \mathbb{Z}$ in (6) completely determine the ECA rules.

As examples we list here a few algebraic expressions of some interesting ECA rules.

$$\text{Rule 30: } F(x)_i = x_{i-1} + x_i + x_{i+1} - 2x_{i-1}x_i - x_ix_{i+1} - 2x_{i-1}x_{i+1} + 2x_{i-1}x_ix_{i+1}$$

$$\text{Rule 90: } F(x)_i = x_{i-1} + x_{i+1} - 2x_{i-1}x_{i+1}$$

$$\text{Rule 108: } F(x)_i = x_i + x_{i-1}x_{i+1} - 2x_{i-1}x_ix_{i+1}$$

$$\text{Rule 110: } F(x)_i = x_i + x_{i+1} - x_ix_{i+1} - x_{i-1}x_ix_{i+1}$$

$$\text{Rule 184: } F(x)_i = x_{i-1} - x_{i-1}x_i + x_ix_{i+1}$$

$$\text{Rule 232: } F(x)_i = x_{i-1}x_i + x_ix_{i+1} + x_{i-1}x_{i+1} - 2x_{i-1}x_ix_{i+1}$$

Note how simple for example the algebraic expression of the "complex" ECA rule 110 is.

As is well known the ECA rule space can be partitioned into 88 equivalence classes, because ECA rules are equivalent under the symmetry operations of exchanging left/right and 0/1 complementation. For the local function $f(x)_i = f(x_{i-1}, x_i, x_{i+1})$ these symmetry operations are given by $T^{left/right}(f(x)_i) = f(x_{i+1}, x_i, x_{i-1})$ and $T^{0/1}(f(x)_i) = 1 - f(1 - x_{i-1}, 1 - x_i, 1 - x_{i+1})$.

For example, for ECA rule 110 the equivalent rules are:

$$\text{Rule 110: } F(x)_i = x_i + x_{i+1} - x_ix_{i+1} - x_{i-1}x_ix_{i+1}$$

$$\text{Rule 137: } F(x)_i = 1 - x_{i-1} - x_i - x_{i+1} + x_{i-1}x_i + 2x_ix_{i+1} + x_{i-1}x_{i+1} - x_{i-1}x_ix_{i+1}$$

$$\text{Rule 124: } F(x)_i = x_{i-1} + x_i - x_{i-1}x_i - x_{i-1}x_ix_{i+1}$$

$$\text{Rule 193: } F(x)_i = 1 - x_{i-1} - x_i - x_{i+1} + 2x_{i-1}x_i + x_ix_{i+1} + x_{i-1}x_{i+1} - x_{i-1}x_ix_{i+1}$$

From now on we will refer to such a group of equivalent ECA rules solely by their lowest decimal ECA number. For example, by referring to ECA rule 110 we actually refer to the four rules $\{110, 137, 124, 193\}$.

Note that the approach developed here can be extended in various ways, for example to one-dimensional CA with state space $\{0, 1\}$ but larger neighbourhood or to two-dimensional CA with state space $\{0, 1\}$, etc.

5 Classification of Elementary Cellular Automata

We will now classify ECA in regard to their topological dynamics properties, that is according to the scheme introduced by Gilman [9] and modified by Kurka [10].

First, we need some more symbolic dynamics definitions and notions. A *word* u is a finite symbolic sequence $u = u_0 \dots u_{l-1}$, with $u_i \in S$, where S is a finite *alphabet*, e.g. in the case of ECA the state set $\{0, 1\}$. The length of u is denoted by $l = |u|$. The set of words of S of length l is denoted by S^l , the set of all words of S with $l > 0$ is S^+ . The *cylinder set* $[u]_0$ of u consists of all points $x \in S^{\mathbb{Z}}$ whose initial part is u , i.e. $[u]_0 = \{x \in S^{\mathbb{Z}} : x_{[0,l)} = u\}$.

A word $u \in S^+$ with $|u| \geq m, m > 0$, is *m-blocking* for an one-dimensional CA F , if there exists an offset $q \in [0, |u| - m]$ such that

$$\forall x, y \in [u]_0, \forall n \geq 0, F^n(x)_{[q, q+m)} = F^n(y)_{[q, q+m)}.$$

See Figure 1. It has been shown that one-dimensional CA, and therefore ECA, are either sensitive or almost equicontinuous, the latter being equivalent to having a blocking word.

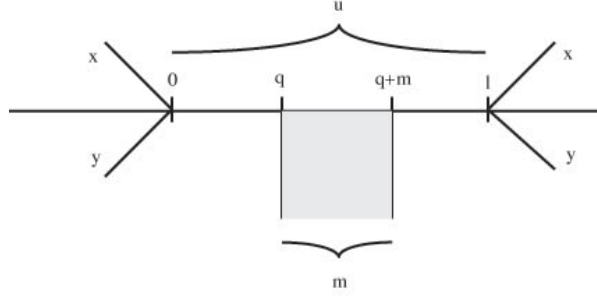


Figure 1: Blocking word.

Proposition 1 (Kurka [11]). *For any one-dimensional CA F with radius $r > 0$ the following conditions are equivalent.*

- (1) F is not sensitive.
- (2) F has an r -blocking word.
- (3) F is almost equicontinuous.

If a configuration x contains a m -blocking word u , then the sequence $x_{[q, q+m)}$, i.e. the states of the cells in the segment $[q, q+m)$, are at all times independent of the initial states outside of the blocking word u . Hence, the following corollary holds.

Corollary 2. *For any one-dimensional CA F with radius $r > 0$ the following conditions are equivalent.*

- (1) F has a m -blocking word with $m \geq r$.
- (2) F has a word $u \in S^+$ with $|u| \geq m, m > 0$ and an offset $q \in [0, |u| - m]$ such that $\forall x \in [u]_0$ the sequence $x_{[q, q+m)}$ is eventually temporally periodic.

Proof. (1) \Rightarrow (2): Denote the sequence $x_{[q, q+m)}$ of a blocking word u that is at all times independent of the initial states outside of u by v . The configuration $x = (u)^\infty$ is spatially periodic and hence eventually temporally periodic. Because the sequence v is independent of the states of the cells outside of u , the sequence v is also eventually temporally periodic.

(2) \Rightarrow (1): The condition (2) says that for all $x \in [u]_0$ there is $t \geq 0$ and $p > 0$

such that $F^{t+p}(x)_{[q,q+m)} = F^t(y)_{[q,q+m)}$. Thus, for all $x, y \in [u]_0$ and all $n \geq 0$ the sequence $F^n(x)_{[q,q+m)} = F^n(y)_{[q,q+m)}$ must be independent of the initial states outside of u , hence the word u is m -blocking. \square

Now, we will look systematically for blocking words for ECA. We know by proposition 1 that whenever a blocking word can be found, the corresponding ECA is almost equicontinuous. By corollary 2, we know that this corresponds to finding a word u that contains a sequence that is eventually temporally periodic, independent of the initial states outside of u . As it turns out, we can thereby effectively determine all almost equicontinuous ECA, because any almost equicontinuous ECA corresponds to a blocking word u for which the length $l = |u|$ is bounded.

Proposition 3. *Each almost equicontinuous ECA has at least one blocking word of length $l \leq 4$.*

Proof. In the following, we look for blocking words, starting with the smallest possible length $l = 1$ and then successively for words of greater length (for a visualisation of the definition of a blocking word see again Figure 1). If a blocking word can be found, one or several almost equicontinuous ECA correspond to it. The ECA rules are specified by a rule table which we denote by $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$. For example, ECA rule 110 is given by the table $(0, 1, 1, 0, 1, 1, 1, 0)$. If an entry in the rule table is left unspecified, the entry can take on either of the two values 0 or 1, e.g. the table $(0, 1, 1, 0, 1, 1, 1, t_7)$ refers to the two ECA rules 110 and 111. If a blocking word can be found, we put the corresponding ECA rule table in a list. A blocking word u and the corresponding rule table is denoted by $t_{u,p} = (t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where p is the period with which the eventually periodic sequence in the word u (i.e. the sequence $x_{[q,q+m)}$ referred to in corollary 2) is repeated. For example, $t_{00,1} = (t_0, t_1, t_2, 0, t_4, t_5, 0, 0)$ refers to the blocking word 00 of period $p = 1$ that generates 2^5 ECA rules, as denoted by the corresponding rule table. If a blocking word is found that corresponds to ECA rules which are generated by a rule table obtained by a blocking word already in the list (hence of smaller length), the word and its corresponding rule table is not listed. We also do not list blocking words and their corresponding rule tables if they generate ECA rules that are equivalent to ECA rules generated by a blocking word already in the list.

Let us further assume the following notation: The variable c_i always denotes the states of cells i of a blocking word u that are at all times independent of the initial states of the cells outside of the blocking word u . The variable x_i on the other hand denotes the states of cells i that are in principle influenceable by the initial states of the cells outside of u . The state x_i of such a cell i is left undetermined, i.e. the value can either be 0 or 1. If it is known for configurations $x, y \in [u]_0$ that the states x_i and y_i of some cell i differ, we write \bar{x}_i . For example, the “scenario” $\begin{array}{ccc} \bar{x}_{-1} & c_0 & \bar{x}_1 \\ x_{-1} & c_0 & x_1 \end{array}$ refers to two configurations $x, y \in [u]_0$ that share the blocking word $u = c_0$ of length $l = 1$ that is repeated

The proof has two parts. In part A, we determine all blocking words of length $l \leq 4$. In part B, we show that for any blocking word u of length $l > 4$ there is a corresponding blocking word of length $l \leq 4$.

(a) With $l = 1$, the following scenarios are possible: (1) $\begin{matrix} \bar{x}_{-1} & c_0 & \bar{x}_1 \\ x_{-1} & c_0 & x_1 \end{matrix}$,

for one i , $c'_i \neq c_i$. Note that there are further scenarios possible that however do not yield further valid rule tables and are not listed here. Scenario (1) yields the rule table $t_{1,1} = (1, 1, t_2, t_3, 1, 1, t_6, t_7)$ (and the table $t_{0,1} = (t_0, t_1, 0, 0, t_4, t_5, 0, 0)$, but as said, tables that generate ECA rules equivalent to already generated rules are not listed). Scenarios (2), (3) and (4) do not yield rule tables that generate ECA rules not already listed. For scenario (5),

where $c'_0 \neq c_0$. The first case generates no new ECA rules. The second case yields the rule table $t_{1,2} = (0, 0, 1, 1, 0, 0, 1, 1)$. Scenarios (6) and (7) yield rule tables already listed. Scenario (8) yields $t_{1,2} = (0, 0, 0, 0, 0, 0, 0, 1)$.

and (2) $\begin{matrix} \bar{x}_{-1} & c_0 & c_1 & \bar{x}_2 \\ x_{-1} & c'_0 & c'_1 & x_2 \end{matrix}$, where $c'_i \neq c_i$. Scenarios (1) and (2) yield the

(c) For $l = 3$, there are basically again scenarios in analogy to the cases (a) and (b) possible. The scenarios that effectively generate rule tables not

where $c'_1 \neq c_1$. Scenario (1) yields the blocking words and rule tables $t_{010,1} = (t_0, t_1, 0, 0, t_4, 1, 0, t_7)$ and $t_{101,1} = (t_0, 1, 0, t_3, 1, 1, t_6, t_7)$. Scenario (2) yields

$t_{000,2} = (0, 0, t_2, 0, 0, 0, 0, 1)$. Note that e.g. the scenario

\bar{x}_{-1}	c_0	c_1	c_2	\bar{x}_3
x_{-1}	\bar{x}_0	c_1	\bar{x}_2	x_3
	c_0	c_1	c_2	

does not generate new rule tables.

(d) For $l = 4$, there are basically again scenarios in analogy to the cases (a), (b) and (c) possible. Effectively, the only scenario that leads to a blocking word generating a rule table not yet listed is

\bar{x}_{-1}	c_0	c_1	c_2	c_3	\bar{x}_4
x_{-1}	c_0	c_1	c_2	c_3	x_4

, yielding

the rule table $t_{0110,1} = (t_0, 1, 0, 0, 1, t_5, 0, t_7)$. Note again that e.g. the scenario

\bar{x}_{-1}	c_0	c_1	c_2	c_3	\bar{x}_4
x_{-1}	\bar{x}_0	c'_1	c'_2	\bar{x}_3	x_4
	c_0	c_1	c_2	c_3	

, where at least for one i $c'_i \neq c_i$, does not lead to new rule tables.

With this we conclude Part A. Let us summarise the blocking words and the corresponding rule tables found so far:

$$\begin{aligned}
t_{0,1} &= (t_0, t_1, 0, 0, t_4, t_5, 0, 0) \\
t_{1,2} &= (0, 0, 1, 1, 0, 0, 1, 1) \\
t_{1,2} &= (0, 0, 0, 0, 0, 0, 0, 1) \\
t_{00,1} &= (t_0, t_1, t_2, 0, t_4, t_5, 0, 0) \\
t_{01,1} &= (t_0, t_1, 0, t_3, 1, 1, 0, t_7) \\
t_{10,1} &= (t_0, 1, 0, 0, t_4, 1, t_6, t_7) \\
t_{00,2} &= (0, 0, t_2, 1, 0, t_5, 1, 1) \\
t_{01,2} &= (t_0, 0, 1, 1, 0, 0, 1, t_7) \\
t_{010,1} &= (t_0, t_1, 0, 0, t_4, 1, 0, t_7) \\
t_{101,1} &= (t_0, 1, 0, t_3, 1, 1, t_6, t_7) \\
t_{000,2} &= (0, 0, t_2, 0, 0, 0, 0, 1) \\
t_{0110,1} &= (t_0, 1, 0, 0, 1, t_5, 0, t_7)
\end{aligned}$$

Part B: In the general case, i.e. for $l > 4$, we can conclude in analogy to the cases already considered, i.e. the cases with $l \leq 4$, that the following scenarios could possibly lead to new blocking words:

$$\begin{aligned}
(1) & \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\ x_{-1} & c_0 & c_1 & \dots & c_{l-2} & c_{l-1} & x_l \end{pmatrix}, (2) \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\ c_{-1} & c_0 & c_1 & \dots & c_{l-2} & c_{l-1} & c_l \end{pmatrix}, \\
(3) & \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & & & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\ & & & & \bar{x}_{q-1} & c_q & \dots & c_{q+m-1} & \bar{x}_{q+m} \\ & & & & x_{q-1} & c_q & \dots & c_{q+m-1} & x_{q+m} \end{pmatrix}, \\
(4) & \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & & & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\ & & & & \bar{x}_{q-1} & c_q & \dots & c_{q+m-1} & \bar{x}_{q+m} \\ & & & & c_{q-1} & c_q & \dots & c_{q+m-1} & c_{q+m} \end{pmatrix}, \\
(5) & \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\ x_{-1} & c'_0 & c'_1 & \dots & c'_{l-2} & c'_{l-1} & x_l \end{pmatrix}, (6) \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\ c'_{-1} & c'_0 & c'_1 & \dots & c'_{l-2} & c'_{l-1} & c'_l \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(7) & \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\ & & & & \dots & & & & \\ & & \bar{x}_{q-1} & c_q & \dots & c_{q+m-1} & \bar{x}_{q+m} & & \\ & & x_{q-1} & c'_q & \dots & c'_{q+m-1} & x_{q+m} & & \end{pmatrix}, \\
(8) & \begin{pmatrix} \bar{x}_{-1} & c_0 & c_1 & \dots & & \dots & & c_{l-2} & c_{l-1} & \bar{x}_l \\ & & & & \dots & & & & & \\ & & \bar{x}_{q-1} & c_q & c_{q+1} & \dots & c_{q+m-2} & c_{q+m-1} & \bar{x}_{q+m} & \\ & & \bar{x}_q & c'_{q+1} & \dots & c'_{q+m-2} & \bar{x}_{q+m-1} & & & \\ & & c_q & c_{q+1} & \dots & c_{q+m-2} & c_{q+m-1} & & & \end{pmatrix},
\end{aligned}$$

with $m \geq 1$ and where at least for one i , $c'_i \neq c_i$.

Case (1) yields blocking words already listed, because for $l > 4$ the conditions to be satisfied in order to obtain a blocking word u are entailed in the conditions to obtain a blocking word u with $l \leq 4$. The same reasoning applies to cases (2), (3) and (4). The basic reason that such a reduction is possible is due to the fact that the conditions to be satisfied in order to obtain a blocking word depend on the values of the boundary cells, here the values \bar{x}_{-1} and \bar{x}_l (respectively the values \bar{x}_{q-1} and \bar{x}_{q+m} in cases (3) and (4)), but not on the values of the cells to the left (of $i = -1$) and right (of $i = l$) of the boundary cells, as can be checked with the scenarios treated in Part A.

Let us then look closer at case (5). We will show that if there is a blocking word $c_1 c_2 \dots c_{l-2} c_{l-1}$, the word is repeated with period $p = 2$, because if the word is blocking, the word at the next time step (in case (5)) must be $\bar{c}_1 \bar{c}_2 \dots \bar{c}_{l-2} \bar{c}_{l-1}$. The bar signifies that the state c_i of the cell i must change, i.e. $\bar{c}_i = (1 - c_i)$. Without loss of generality, we can consider only the 2^4 boundary conditions for blocking words at successive time steps. That is, given the word $c_1 c_2 \dots c_{l-2} c_{l-1}$, we consider at the next time-step all the $(2^4 - 2)$ possible cases: $c_1 c_2 \dots c_{l-2} \bar{c}_{l-1}$, $c_1 c_2 \dots \bar{c}_{l-2} c_{l-1}$, etc., excluding the two cases $c_1 c_2 \dots c_{l-2} c_{l-1}$ and $\bar{c}_1 \bar{c}_2 \dots \bar{c}_{l-2} \bar{c}_{l-1}$. It suffices to consider the case $c_1 c_2 \dots c_{l-2} \bar{c}_{l-1}$. The other cases can be dealt with analogously. The temporal evolution of the ECA generates in this case the following scheme:

$$\begin{array}{ccccccc}
\bar{x}_{-1} & c_0 & c_1 & \dots & c_{l-2} & c_{l-1} & \bar{x}_l \\
x_{-1} & c_0 & c_1 & \dots & c_{l-2} & \bar{c}_{l-1} & x_l \\
x_{-1} & c_0 & c_1 & \dots & \bar{c}_{l-2} & c_{l-1}^2 & x_l \\
& & & \dots & & & \\
x_{-1} & c_0 & \bar{c}_1 & \dots & c_{l-2}^{l-2} & c_{l-1}^{l-2} & x_l \\
x_{-1} & \bar{c}_0 & c_1^{l-1} & \dots & c_{l-2}^{l-1} & c_{l-1}^{l-1} & x_l
\end{array}$$

The superscript denotes the time-step n . The third, fifth and sixth line are due to the fact that if the state of e.g. the cell $l - 2$ at time step $n = 2$ would not change, one would obtain a blocking word of shorter length $(l - 1)$. By checking all 2^4 possible values for the boundary states of the initial word $c_1 c_2 \dots c_{l-2} c_{l-1}$ it can be shown that the above scheme cannot be satisfied. Thus, any initial word $c_1 c_2 \dots c_{l-2} c_{l-1}$ evolves in the next time step into either the word $c_1 c_2 \dots c_{l-2} c_{l-1}$ or the word $\bar{c}_1 \bar{c}_2 \dots \bar{c}_{l-2} \bar{c}_{l-1}$. In the first case, a blocking word of period $p = 1$ is generated, in the second case, i.e. for $p = 2$, one can find a blocking word of

length $l = 2$, as can easily be shown.

The case (6) can be reduced to the case already treated under (a (8)) in Part A, the case (7) to the case (5) and the case (8) again to the case treated under (c (2)) (or the example in (d) respectively) in Part A.

With this we conclude our analysis. In Part A, we have identified all blocking words of length $l \leq 4$. For $l \geq 2$, we omitted, for reasons of space, the presentation of the cases that do not lead to blocking words or to blocking words already identified. In Part B, we have concluded from the cases for $l \leq 4$ on the general form of the scenarios that could possibly lead to blocking words for $l > 4$. These general scenarios could then be reduced to the scenarios obtained for $l \leq 4$. One case (case (5)) required a separate treatment and was analysed by means of an example.

To arrive at a complete list of blocking words for $l \leq 4$ and to exclude additional blocking words for $l > 4$, great care and efforts have been invested. Moreover, the completeness of the list was corroborated by extensively sampling the space of initial configurations, which yielded no additional blocking words. An explicit check of the completeness of the cases considered in our analysis would require the help of a computer, performing the analysis presented above on a case-by-case basis. Alternatively, to demonstrate the impossibility of additional blocking words, the systems of equations generated from the conditions for blocking words and the algebraic expressions of ECA rules could be used, systematically evaluated for each single case. □

The proof of proposition 3 allows to give, for ECA, a stronger version of proposition 1. Let us call a word u of length l *invariant* for an ECA F , if for all $x \in [u]_0$, there is a $p > 0$ such that $F^p(x)_{[0,l]} = x_{[0,l]}$.

Corollary 4. *An ECA F is almost equicontinuous if and only if F has an invariant word.*

Proof. See the proof of proposition 3. □

Proposition 3 (or corollary 4 respectively) allows to determine the almost equicontinuity of all ECA rules. The following list gives the invariant words of shortest length and the almost equicontinuous ECA rules corresponding to them.

Corollary 5. *Invariant words of period $p = 1$ and corresponding ECA rules:*

0: 0, 4, 8, 12, 72, 76, 128, 132, 136, 140, 200, 204.

00: 32, 36, 40, 44, 104, 108, 160, 164, 168, 172, 232.

01: 13, 28, 29, 77, 156.

10: 78.

010: 5.

101: 94.

0110: 73.

Invariant words of period $p = 2$ and corresponding ECA rules:

1: 1.

0: 51.
 00: 19, 23.
 01: 50, 178.
 000: 33.

Conversely, we now also know the sensitive ECA rules.

Proposition 6. *The following rules are sensitive:*

2, 3, 6, 7, 9, 10, 11, 14, 15, 18, 22, 24, 25, 26, 27, 30, 34, 35, 37, 38, 41, 42, 43, 45, 46, 54, 56, 57, 58, 60, 62, 74, 90, 105, 106, 110, 122, 126, 130, 134, 138, 142, 146, 150, 152, 154, 162, 170, 184.

Proof. Follows from Proposition 1, 3 and corollary 5. \square

The class of sensitive ECA is large, because in the Cantor space left- or right-shifting rules are sensitive. We return to this point later.

From the almost equicontinuous ECA rules, we can further specify the *equicontinuous* ones. We use the following lemma.

Lemma 7 (Kurka [11]). *A one-dimensional almost equicontinuous CA F is either equicontinuous or almost equicontinuous but not equicontinuous. It is equicontinuous if and only if:*

(1) *There exists a preperiod $m \geq 0$ and a period $p > 0$, such that $F^{m+p} = F^m$.*

It is almost equicontinuous but not equicontinuous if and only if:

(2) *There is at least one point $x \in X$ for which the almost equicontinuous CA F is not equicontinuous.*

Proposition 8. *The following rules are equicontinuous:*

0, 1, 4, 5, 8, 12, 19, 29, 36, 51, 72, 76, 108, 200, 204.

Proof. The proof is by showing that condition (1) of Lemma 7 holds. We only give an example for a specific ECA rule.

Rule 72 is equicontinuous with preperiod $m = 2$ and period $p = 1$, because, by using the algebraic expression for the local function, we obtain

$$\begin{aligned} F(x)_i &= x_{i-1}x_i + x_ix_{i+1} - 2x_{i-1}x_ix_{i+1} \\ F^2(x)_i &= x_{i-1}x_i - x_{i-2}x_{i-1}x_i + x_ix_{i+1} - 2x_{i-1}x_ix_{i+1} + x_{i-2}x_{i-1}x_ix_{i+1} - x_ix_{i+1}x_{i+2} + \\ &\quad x_{i-1}x_ix_{i+1}x_{i+2} \\ F^3(x)_i &= x_{i-1}x_i - x_{i-2}x_{i-1}x_i + x_ix_{i+1} - 2x_{i-1}x_ix_{i+1} + x_{i-2}x_{i-1}x_ix_{i+1} - x_ix_{i+1}x_{i+2} + \\ &\quad x_{i-1}x_ix_{i+1}x_{i+2}. \end{aligned}$$

Hence, $F^3(x)_i = F^2(x)_i, \forall i \in \mathbb{Z}$. Thus, $F^3 = F^2$. \square

Proposition 9. *The following rules are almost equicontinuous but not equicontinuous:*

13, 23, 28, 32, 33, 40, 44, 50, 73, 77, 78, 94, 104, 128, 132, 136, 140, 156, 160, 164, 168, 172, 178, 232.

Proof. The proof is by showing that condition (2) of Lemma 7 holds. We only give an example for a specific ECA rule.

ECA rule 104 is almost equicontinuous but not equicontinuous, because $(10)^\infty$ is not an equicontinuous point.

Assume the configuration $x = (10)^\infty$ and an integer $q > 0$ such that

$$\forall y \in X, (x_{[-q,q]} = y_{[-q,q]}) \Rightarrow (d(x, y) < 2^{-q}).$$

Assume that y differs from x at cells $(-q-1)$ and $(q+1)$, that is $y_{-q-1} = 1 - x_{-q-1}$ and $y_{q+1} = 1 - x_{q+1}$. Then, as can easily be shown by using the algebraic expression of ECA rule 104,

$$d(F^n(x), F^n(y)) < 2^{-(q-n)}$$

for all $n \geq 0$. Hence, ECA 104 is not equicontinuous at the point $x = (10)^\infty$. \square

From the sensitive ECA, we can distinguish further the *positively expansive* ECA.

First, we need the definition of *permutivity* for ECA [19]. An ECA F is *left-permutive* if $(\forall u \in S^2), (\forall b \in S), (\exists! a \in S): f(au) = b$. It is *right-permutive* if $(\forall u \in S^2), (\forall b \in S), (\exists! a \in S): f(ua) = b$. The ECA F is *permutive* if it is either left-permutive or right-permutive.

Then, we need the following lemma.

Lemma 10 (Kurka [11]). *A one-dimensional CA F is positively expansive if the following condition holds.*

(1) *The CA is both left- and right-permutive.*

A one-dimensional sensitive CA F is not positively expansive if and only if the following condition holds.

(2) *There is no $\epsilon > 0$ such that for all $x \neq y \in X$ there is $n \geq 0$ with $d(F^n(x), F^n(y)) \geq \epsilon$.*

Proposition 11. *The following ECA rules are sensitive but not positively expansive:*

2, 3, 6, 7, 9, 10, 11, 14, 15, 18, 22, 24, 25, 26, 27, 30, 34, 35, 37, 38, 41, 42, 43, 45, 46, 54, 56, 57, 58, 60, 62, 74, 106, 110, 122, 126, 130, 134, 138, 142, 146, 152, 154, 162, 170, 184.

Proof. The proof is by showing that condition (2) of Lemma 10 holds. We only give an example for a specific ECA rule.

ECA rule 110 is sensitive but not positively expansive. Assume the expansivity constant $\epsilon = 2^{-m}$, then

$$\forall x \neq y \in X \Rightarrow \exists n \geq 0, F^n(x)_{[-m,m]} \neq F^n(y)_{[-m,m]} \quad (7)$$

must hold. Assume the configuration $x = (00110111110001)^\infty$ and an integer $q > 0$ such that $14q > m$. Then, for a configuration $y \in X$ that differs from x at the cells $14q, 14q+1, 14q+2$, (7) does not hold. \square

Proposition 12. *The following ECA rules are positively expansive: 90, 105, 150.*

Proof. For ECA rules 90, 105 and 150 condition (1) of Lemma 10 holds. \square

For ECA left- and right-permutivity is equivalent to positive expansivity.

Proposition 13. *ECA are positively expansive if and only if they are both left- and right-permutive.*

Proof. Follows from Proposition 6, 11 and 12. \square

Note that Proposition 13 does not hold generally for one-dimensional CA [11].

We summarize the findings of this section in the following table, which shows all ECA rules according to whether they have the property of equicontinuity, almost equicontinuity, sensitivity or positively expansivity.

almost equicontinuous		sensitive	
equicontinuous			positively expansive
0, 1, 4, 5, 8, 12, 19, 29, 36, 51, 72, 76, 108, 200, 204	13, 23, 28, 32, 33, 40, 44, 50, 73, 77, 78, 94, 104, 128, 132, 136, 140, 156, 160, 164, 168, 172, 178, 232	2, 3, 6, 7, 9, 10, 11, 14, 15, 18, 22, 24, 25, 26, 27, 30, 34, 35, 37, 38, 41, 42, 43, 45, 46, 54, 56, 57, 58, 60, 62, 74, 106, 110, 126, 130, 134, 138, 142, 152, 154, 162, 170, 184	90, 105, 150

6 Classification of Sensitive Elementary Cellular Automata

The classification of the degree of "complexity" manifest in the space-time patterns generated by ECA rules has been in the center of CA research for many years. It is intuitively clear that the sensitivity property is a source of the apparent "complexity" of ECA behaviour. It would therefore be beneficial to have a finer classification of the sensitive ECA rules.

We classify sensitive rules further according to whether they are eventually weakly periodic and/or surjective. The surjectivity property is actually sufficient to determine which ECA are chaotic (in the sense of Devaney [20]). This fact has already been demonstrated by Cattaneo et al. [21]. For the sake of completeness, we rederive the result here and propose the further subclass of the "eventually weakly periodic" and sensitive ECA.

A configuration x is called *weakly periodic*, if there is $q \in \mathbb{Z}$ and $p > 0$ such that $F^p \sigma^q(x) = x$ [19]. We define a configuration x as *eventually weakly periodic* if there is $q \in \mathbb{Z}$ and $n, p > 0$ such that $F^{n+p} \sigma^q(x) = F^n(x)$. We call an ECA *eventually weakly periodic*, if the ECA is not eventually periodic, but for all configurations x eventually weakly periodic.

Proposition 14. *The following sensitive ECA rules are eventually weakly periodic:*

2, 3, 10, 11, 15, 24, 34, 38, 42, 46, 138, 170.

Proof. We demonstrate the proof by an example for a specific ECA rule.

Employing the algebraic expression for ECA rule 10, it can easily be shown that $F^2 \sigma^{-1}(x) = F(x)$ for all configurations x .

Hence, ECA rule 10 is eventually weakly periodic with $n = 1, p = 1$ and $q = -1$. \square

The classification of eventually weakly periodic ECA maps is not complete. There might be ECA which are eventually weakly periodic, but with such large n or p that prevents calculating the forward orbits as easily as in the proof of Proposition 14.

We now look at the chaoticity of ECA maps. According to the often used definition of (topological) chaos by Devaney [20], a map $F : X \rightarrow X$ is chaotic if F is sensitive, transitive and if the set of periodic points of F is dense in X .

First, we study the surjectivity of ECA maps. This is actually sufficient to establish transitivity of F and the density of periodic points in X under F .

A CA is surjective if and only if it has no *Garden-of-Eden* configurations, that is configurations which have no pre-image. A necessary (but not sufficient) condition for surjectivity is that the local rule is balanced [11]. For ECA rules this means that the local rule table contains 4 zeros and 4 ones. Further, any permutive CA is surjective [11].

Proposition 15. *The following ECA rules are surjective:*

15, 30, 45, 51, 60, 90, 105, 106, 150, 154, 170, 204.

Proof. Apart from rule 51 and rule 204, the above listed rules are permutive, hence surjective. Rule 51 and rule 204 are surjective, because they are, trivially, bijective.

For the ECA rules that are not listed, but satisfy the balance condition, it can be shown that they possess Garden-of-Eden configurations. For example, ECA rule 184 satisfies the balance condition, nevertheless it is not surjective, because any configuration containing pattern (1100) is a Garden-of-Eden as can easily be shown. \square

Next, we show that for ECA transitivity is equivalent to permutivity. An one-dimensional CA F is transitive if for any nonempty open sets, $U, V \subseteq X$ there exists $n > 0$ with $F^n(U) \cap V \neq \emptyset$.

Proposition 16. *A ECA is transitive if and only if it is permutive.*

Proof. Transitivity of one-dimensional CA implies its surjectivity and sensitivity [11]. From Proposition 6 and 15 we gain that ECA that are surjective and sensitive are permutive. For permutive ECA either of the following two condition holds:

- (1) for all open sets $U \in X$ there is $n > 0$ such that $F^n(U)$ covers all of X ,
- (2) the value of the local function $f(x_{i-1}, x_i, x_{i+1})$ is determined by either the value x_{i-1} or x_{i+1} .

If (2) does not hold, there is $n > 0$ such that $F^n(U)$ covers, due to the permutivity of F , all of X . Hence, F is in this case transitive. If (2) does hold, the local function f is given by the rule tables $(1, 0, 1, 0, 1, 0, 1, 0)$ or $(0, 0, 0, 0, 1, 1, 1, 1)$, i.e. the rules 170 or 15 (equivalent rules included). Rule 170 (15) is the shift-map $F(x)_i = x_{i+1}$ (for ECA rule 15: $F(x)_i = 1 - x_{i-1}$), hence transitive. \square

Corollary 17. *A ECA map is transitive if and only if it is surjective and sensitive.*

Proof. Follows from Proposition 6, 15 and 16. \square

Next, we show that for ECA surjectivity implies that the set of periodic points of F is dense in X .

Proposition 18. *Surjective ECA have a dense set of periodic points in X .*

Proof. Surjective ECA are either almost equicontinuous or sensitive. Almost equicontinuous one-dimensional CA that are surjective have a dense set of periodic points [22]. The sensitive ECA that are surjective are permutive and permutive one-dimensional CA are known to have a dense set of periodic points (through the property of closingness [11]). \square

A central open question of CA theory is the conjecture that for one-dimensional CA surjectivity implies a dense set of periodic points. For ECA the conjecture is answered in the positive.

Hence, for ECA transitivity, or permutivity, implies chaos.

Corollary 19. *The following ECA rules are chaotic in the sense of Devaney: 15, 30, 45, 60, 90, 105, 106, 150, 154, 170.*

Figure 2 summarises these results.

7 Discussion

The results of this paper show that one can classify the dynamic behaviour of every elementary cellular automata (ECA) in terms of the standard notions of dynamical system theory, that is according to the classification proposed by Gilman [9] and Kurka [10]. We also determined which ECA are chaotic in the sense of Devaney, rederiving a result by Cattaneo et al. [21]. This gives a fairly complete picture of the dynamical system properties of ECA in the standard topology, as summarised in Fig. 2. The topological dynamics approach to CA

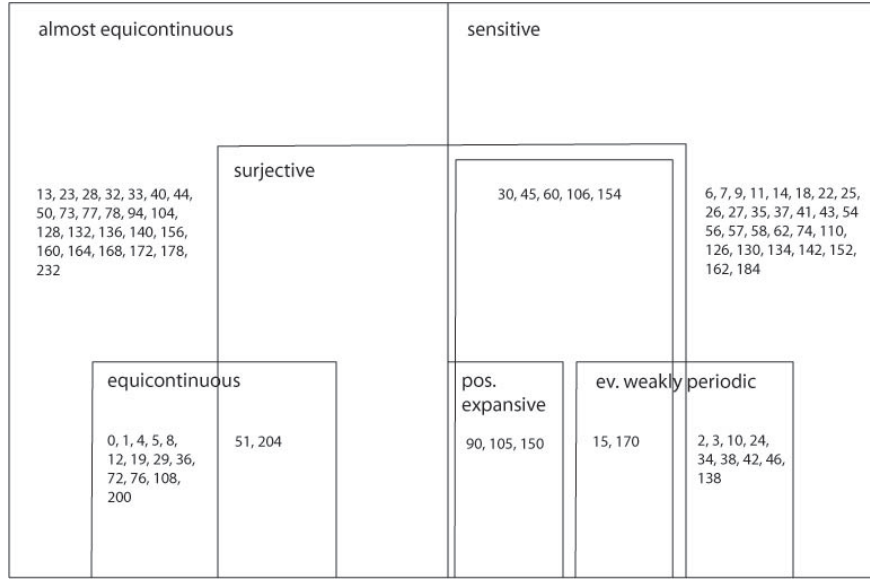


Figure 2: Classification diagram for the elementary cellular automata (ECA). The chaotic ECA are inside the double-framed box.

thus delivers a relevant and coherent account of the dynamical behaviour of ECA.

In the light of our results, the class of "complex" ECA can be characterised as the class of ECA that are sensitive, but not surjective and not eventually weakly periodic. This class corresponds well to what one would intuitively regard as "complex" given the space-time patterns of ECA. In particular, the ECA rules of Wolfram's class (W4) seem to fall into this class.

Among the ECA rules, a few deserve special interest from a computational point of view. The most prominent example is ECA rule 110 which has been shown to be computationally universal [23]. Based on our results we conjecture that sensitivity is a necessary condition of computational universality. In contrast, Wolfram conjectured that, for example, ECA rule 73, which is not sensitive, may be computationally universal [18]. This difference is due to the fact that our results hold generally for ECA without any restrictions on the initial conditions, whereas Wolfram considers specific sets of initial configurations on which the rule acts. On such a restricted set of configurations, ECA rule 73 might indeed be sensitive.

If a CA is sensitive, then its dynamics defies numerical computation for practical purposes, because a finite precision computation of an orbit may result in a completely different orbit than the real orbit. Hence, while sensitivity seems inherent to the in principle computationally most powerful rules, as e.g. rule 110, their limited robustness to small changes in the initial conditions may

impair their practical usage in a physical or biological system: Even a single bit-flip in the input of a sensitive ECA may completely change the computed output.

Among the many questions left open, a natural extension of our study would consist in giving a complete characterisations in the topological dynamics sense for more general CA than ECA. Examples by Cattaneo et al. [21], however, show that the approach taken here to establish chaoticity can already fail in slightly more general settings. In the general case, long-term properties of CA and hence classification schemes based on these properties are typically undecidable. It would therefore be useful to pinpoint where exactly undecidability enters.

Establishing a verifiable notion of computational universality in the Turing-machine sense in terms of necessary and sufficient conditions related to the dynamic behaviour of the underlying system would greatly advance our understanding of the relation between computational and dynamic properties of physical and biological systems. Part of the problem to clarify this relation is that there is no unanimous accepted definition of computational universality for computational systems such as CA (See e.g. the discussion by Ollinger [24] and Delvenne et al. [25]. Delvenne et al. also prove necessary conditions for a symbolic system to be universal, according to their definition of universality, and demonstrate the existence of an universal and chaotic system on the Cantor space.). To different definitions of universality, there might thus correspond different topological dynamics properties. Despite this fact, we conjecture that for ECA sensitivity and non-surjectivity are necessary conditions of universality. This conjecture is in accordance with the intuitive idea that systems at the "edge of chaos", i.e. systems with neither too simple nor chaotic dynamical behaviour, are the computationally relevant systems for biology. Such intermittent systems have, moreover, been characterised as having the largest complexity in the sense that their behaviour is the hardest to predict [27]. If computation is measured as a reduction of complexity [26], the intermittent systems may then be said to provide the complexity needed for efficient computations.

The extension of the results and observations from ECA to general one-dimensional CA or higher-dimensional CA is thus not without problems. Being much more tractable, ECA provide an important benchmark to test ideas on universality, the "edge of chaos" hypothesis and, generally, on how "computation" occurs in nature.

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